Discrete signals

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- We consider a discrete and finite time index set $\Rightarrow n = 0, 1, \ldots, N-1 \;\equiv\; [0, N-1]$.
- A discrete signal x is a function mapping the time index set $[0, N-1]$ to a set of real values $x(n)$

$$x : [0, N-1] \rightarrow \mathbb{R}$$

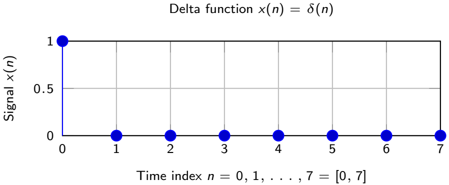
- The values that the signal takes at time index n is $x(n)$
- Sometimes, it makes sense to talk about complex signals $\Rightarrow x : [0, N-1] \rightarrow \mathbb{C}$

$\Rightarrow$ The values $x(n) = x_R(n) + j\, x_I(n)$ the signal takes are complex numbers

- The space of all possible signals is the space of vectors with N components $\Rightarrow \mathbb{R}^N$ (or $\mathbb{C}^N$)

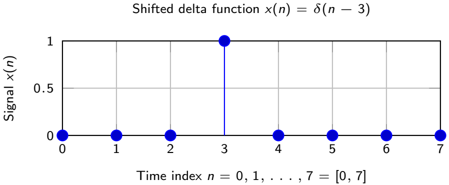
- The discrete delta function $\delta(n)$ is a spike at (initial) time $n = 0$

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{else} \end{cases}$$



- The shifted delta function $\delta(n - n_0)$ has a spike at time $n = n_0$

$$\delta(n - n_0) = \begin{cases} 1 & \text{if } n = n_0 \\ 0 & \text{else} \end{cases}$$

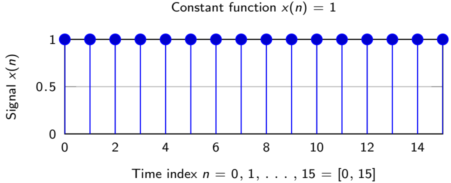


- This is not a new definition. Just a time shift of the previous definition

Constants and square pulses

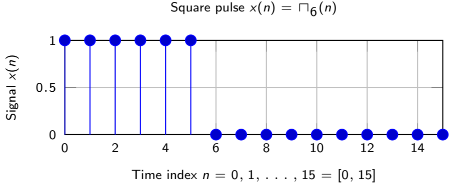
- A constant function $x(n)$ has the same value c for all n

$$x(n) = c, \quad \text{for all } n$$



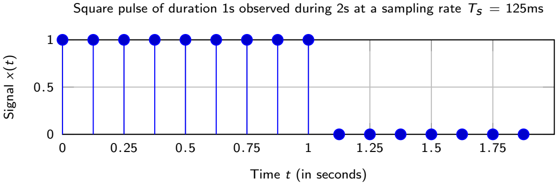
- A square pulse of width M, $\sqcap_M(n)$, equals one for the first M values

$$\sqcap_M(n) = \begin{cases} 1 & \text{if } 0 \leq n < M \\ 0 & \text{if } M \leq n \end{cases}$$



- Can consider shifted pulses $\sqcap_M(n - n_0)$, with $n_0 < N - M$

- The Sampling time T_s is the clock time elapsed between time indexes n and $n+1$
- The sampling frequency $f_s := 1/T_s$ is the inverse of the sampling time
- Discrete time index n represents clock (actual) time $t = nT_s$



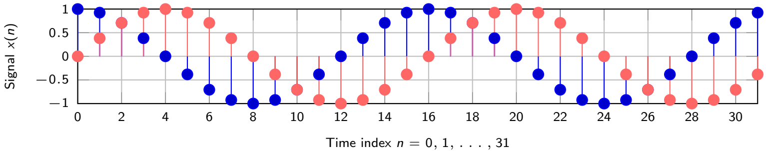
- Total signal duration is $T = NT_s$ $\Rightarrow$ We "hold" the last sample for T_s time units

- For a signal of duration N define (assume N is even):

  $\Rightarrow$ Discrete cosine of discrete frequency k $\Rightarrow x(n) = \cos(2\pi kn/N)$

  $\Rightarrow$ Discrete sine of discrete frequency k $\Rightarrow x(n) = \sin(2\pi kn/N)$

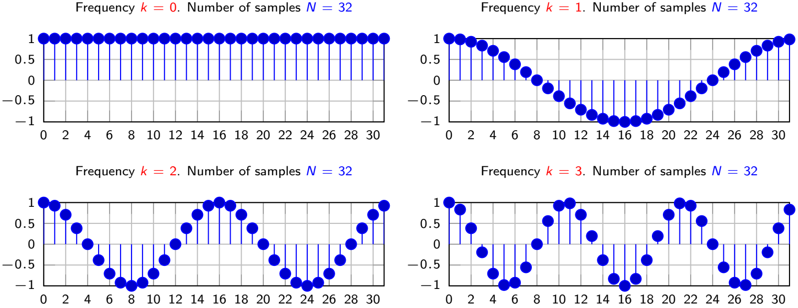
Cosine $x(n) = \cos(2\pi kn/N)$ and sine $x(n) = \sin(2\pi kn/N)$. Frequency $k = 2$ and number of samples $N = 32$.



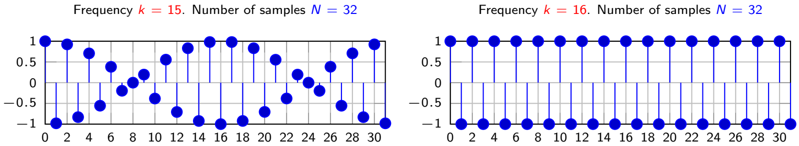
- Frequency k is discrete. I.e., $k = 0, 1, 2, \ldots$

  $\Rightarrow$ Have an integer number of complete oscillations

- Discrete frequency $k = 0$ is a constant
- Discrete frequency $k = 1$ is a complete oscillation
- Frequency $k = 2$ is two oscillations, for $k = 3$ three oscillations ...



- Frequency k represents k complete oscillations
- Although for large k the oscillations may be difficult to see



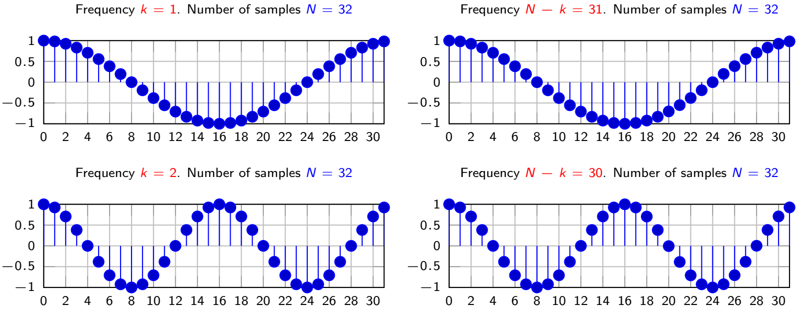
- Do note that we can't have more than $N/2$ oscillations

$\Rightarrow$ Indeed $1 \to -1 \to 1, \to -1, \ldots$

$\Rightarrow$ Frequency $N/2$ is the last one with physical meaning

- Larger frequencies replicate frequencies between $k = 0$ and $k = N/2$

- Frequencies k and $N - k$ represent the same cosine



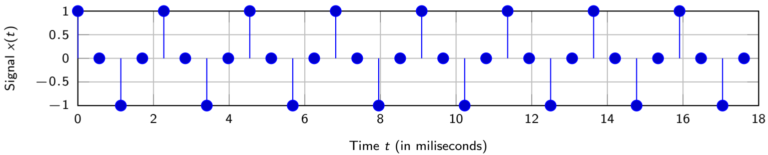
- Actually, if $k + l = \dot{N}$, cosines of frequencies k and l are equivalent
- Not true for sines, but almost. The signals have opposite signs

- What is the discrete frequency k of a cosine of frequency f_0?
- Depends on sampling time T_s, frequency $f_s = \frac{1}{T_s}$, duration $T = NT_s$
- Period of discrete cosine of frequency k is T/k (k oscillations)
- Thus, regular frequency of said cosine is $\Rightarrow f_0 = \dfrac{k}{T} = \dfrac{k}{NT_s} = \dfrac{k}{N} f_s$
- A cosine of frequency f_0 has discrete frequency $k = (f_0/f_s)N$
- Only frequencies up to $N/2 \leftrightarrow f_s/2$ have physical meaning
- Sampling frequency f_s $\Rightarrow$ Cosines up to frequency $f_0 = f_s/2$

- Generate $N = 32$ samples of an A note with sampling frequency $f_s = 1,760\text{Hz}$
- The frequency of an A note is $f_0 = 440\text{Hz}$. This entails a discrete frequency

$$k = \frac{f_0}{f_s}N = \frac{440\text{Hz}}{1,760\text{Hz}}32 = 8$$

The A note observed during $T = NT_s = 18.2\text{ms}$ with a sampling rate $f_s = 1,760\text{Hz}$



- Alternatively $\Rightarrow x(n) = \cos\left[2\pi kn/N\right] = \cos\left[2\pi(f_0/f_s)Nn/N\right]$
- Which simplifies to $\Rightarrow x(n) = \cos\left[2\pi(f_0/f_s)n\right] = \cos\left[2\pi f_0(nT_s)\right]$

Noninteger frequencies

- The frequency k does not need to an integer. In that case we talk of sampled cosines and sines

$\Rightarrow$ Sampled cosine $\Rightarrow x(n) = \cos(2\pi kn/N)$ with arbitrary, not necessarily integer k

$\Rightarrow$ Sampled sine $\Rightarrow x(n) = \sin(2\pi kn/N)$ with arbitrary, not necessarily integer k

- Sampled sines and cosines have fractional oscillations (k not integer)

- Discrete sines and cosines have complete oscillations (k is integer)

$\Rightarrow$ Discrete sines and cosines are used to define Fourier transforms (As we will see later)

- Given two signals x and y with components $x(n)$ and $y(n)$ define the inner product of x and y as

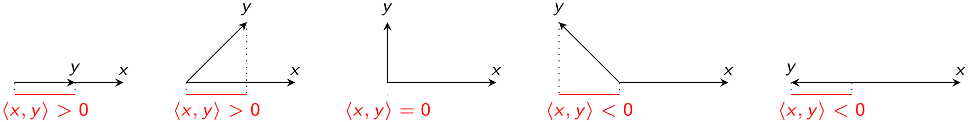
$$\langle x, y\rangle := \sum_{n=0}^{N-1} x(n)y^*(n)$$
$$= \sum_{n=0}^{N-1} x_R(n)y_R(n) - \sum_{n=0}^{N-1} x_I(n)y_I(n) + j\sum_{n=0}^{N-1} x_I(n)y_R(n) + j\sum_{n=0}^{N-1} x_R(n)y_I(n)$$

- This is the same as the inner product between vectors x and y. Just with different notation
- The Inner product is a linear operations $\Rightarrow \langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$
- Reversing the order of the factor results in conjugation $\Rightarrow \langle y, x\rangle = \langle x, y\rangle^*$

- The inner product $\langle x, y\rangle$ is the projection of the signal (vector) y on the signal (vector) x
- The value of $\langle x, y\rangle$ is how much of y falls in x direction

$\Rightarrow$ How much y resembles x. How much x predits y. Knowing x, how much of y we know

$\Rightarrow$ Very importantly, if $\langle x, y\rangle = 0$ the signals are orthogonal. They are "unrelated"



- Define the norm of signal x as $\Rightarrow \|x\| := \left[\sum_{n=0}^{N-1} |x(n)|^2\right]^{1/2} = \left[\sum_{n=0}^{N-1} |x_R(n)|^2 + \sum_{n=0}^{N-1} |x_I(n)|^2\right]^{1/2}$

- Define the energy as the norm squared $\Rightarrow \|x\|^2 := \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{N-1} |x_R(n)|^2 + \sum_{n=0}^{N-1} |x_I(n)|^2$

- The energy of x is the inner product of x with itself $\Rightarrow \|x\|^2 = \langle x, x \rangle$

- Recall that for complex numbers we have $x(n)x^*(n) = |x_R(n)|^2 + |x_I(n)|^2 = |x(n)|^2$

Cauchy Schwarz inequality

- Inner product can't exceed the product of the norms $\Rightarrow\; -\|x\|\,\|y\| \leq \langle x, y\rangle \leq \|x\|\,\|y\|$

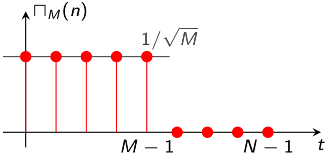
- Inner product squared can't exceed product of energies $\Rightarrow \langle x, y\rangle^2 \leq \|x\|^2\,\|y\|^2$

- If you prefer explicit expressions $\Rightarrow \displaystyle\sum_{n=0}^{N-1} x(n)y^*(n) \leq \left[\sum_{n=0}^{N-1} |x(n)|^2\right]\left[\sum_{n=0}^{N-1} |y(n)|^2\right]$

- The equalities hold if and only if the signals (vectors) x and y are collinear (aligned)

- The unit energy square pulse is the signal $\sqcap_M(n)$ that takes values

$$\sqcap_M(n) = \frac{1}{\sqrt{M}} \qquad \text{if } 0 \leq n < M$$
$$\sqcap_M(n) = 0 \qquad \text{if } M \leq n$$



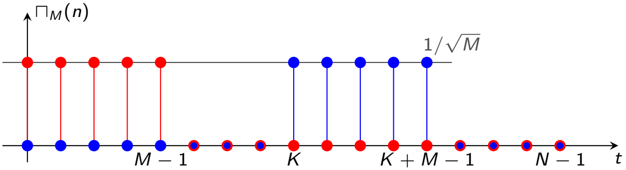
- To compute energy of the pulse we just evaluate the definition

$$\|\sqcap_M\|^2 := \sum_{n=0}^{N-1} |\sqcap_M(n)|^2 = \sum_{n=0}^{M-1} \left|(1/\sqrt{M})\right|^2 = \frac{M}{M} = 1$$

- As name indicates, the unit energy square pulse has unit energy . If pulse height is 1, energy is M.

Shifted pulses

- Shift pulse by modifying argument $\Rightarrow \sqcap_M(n-K)$ $\Rightarrow$ Pulse is now centered at K



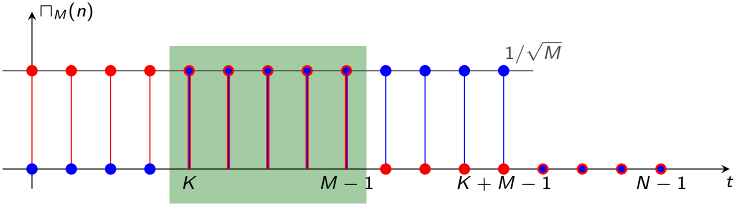
- If the pulse support is disjoint $(K \geq M)$, the inner product of two pulses is zero

$$\langle \sqcap_M(n), \sqcap_M(n-K)\rangle := \sum_{n=0}^{N-1} \sqcap_M(n)\, \sqcap_M\,(n-K) = 0$$

- Pulese are orthogonal $\Rightarrow$ They are "unrelated." One pulse does not predict the other

- If $K < M$ the pulses overlap. They overlap between $n = K$ and $n = M - 1$. Thus, the inner product is

$$\langle \sqcap_M(n), \sqcap_M(n-K) \rangle := \sum_{n=0}^{N-1} \sqcap_M(n) \sqcap_M(n-K) = \sum_{n=K}^{M-1} \left(1/\sqrt{M}\right)\left(1/\sqrt{M}\right) = \frac{M-K}{M} = 1 - \frac{K}{M}$$



- Inner product proportional to relative overlap $\Rightarrow$ How much the pulses are "related" to each other

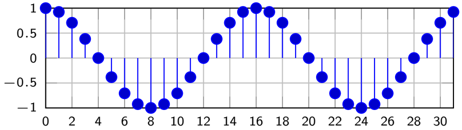
- Discrete complex exponential of discrete frequency k and duration N

$$e_{kN}(n) = \frac{1}{\sqrt{N}}\, e^{j2\pi kn/N} = \frac{1}{\sqrt{N}}\, \exp(j2\pi kn/N)$$

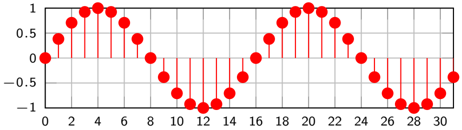
- The complex exponential function is $\Rightarrow e^{j2\pi kn/N} = \cos(2\pi kn/N) + j\sin(2\pi kn/N)$

- The Real part is a discrete cosine. The imaginary part a discrete sine. An oscillation

$\text{Re}\left(e^{j2\pi kn/N}\right)$, with $k = 2$ and $N = 32$



$\text{Im}\left(e^{j2\pi kn/N}\right)$, with $k = 2$ and $N = 32$



[P1] For frequency $k = 0$, the exponential $e_{kN}(n) = e_{0N}(n)$ is a constant $\Rightarrow e_{kN}(n) = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{N}}\, 1$

[P2] For frequency $k = N$, the exponential $e_{kN}(n) = e_{NN}(n)$ is a constant. True for any multiple $k \in \dot{N}$

$$e_{NN}(n) = \frac{e^{j2\pi Nn/N}}{\sqrt{N}} = \frac{(e^{j2\pi})^n}{\sqrt{N}} = \frac{(1)^n}{\sqrt{N}} = \frac{1}{\sqrt{N}}$$

[P3] For $k = \frac{N}{2}$, the exponential $e_{kN}(n) = e_{N/2N}(n) = (-1)^n/\sqrt{N}$. Fastest possible oscillation with N samples

$$e_{N/2N}(n) = \frac{e^{j2\pi(N/2)n/N}}{\sqrt{N}} = \frac{(e^{j\pi})^n}{\sqrt{N}} = \frac{(-1)^n}{\sqrt{N}}$$

That $e^{j2\pi} = 1$ follows from $e^{j\pi} = -1$. Which follows from $e^{j\pi} + 1 = 0$. Relates five most important constants in mathematics.

Theorem

If the frequency difference is $k - l = N$ the signals $e_{kN}(n)$ and $e_{lN}(n)$ coincide for all n, i.e.,

$$e_{kN}(n) = \frac{e^{j2\pi kn/N}}{\sqrt{N}} = \frac{e^{j2\pi ln/N}}{\sqrt{N}} = e_{lN}(n)$$

- Exponentials with frequencies k and l are equivalent if the frequency difference is $k - l = N$

Proof.

- We prove by showing that the ratio $e_{kN}(n)/e_{lN}(n) = 1$. Combine exponents

$$\frac{e_{kN}(n)}{e_{lN}(n)} = \frac{e^{j2\pi kn/N}}{e^{j2\pi ln/N}} = e^{j2\pi(k-l)n/N}$$

- By hypothesis we have that $k - l = N$. Therefore, the latter simplifies to

$$\frac{e_{kN}(n)}{e_{lN}(n)} = e^{j2\pi Nn/N} = \left[e^{j2\pi}\right]^n = 1^n = 1 \qquad \square$$

- Canonical set $\Rightarrow$ Suffice to look at N consecutive frequencies, e.g., $k = 0, 1, \ldots N - 1$

$$\begin{array}{rrrr} -N, & -N+1, & \ldots, & -1 \\ 0, & 1, & \ldots, & N-1 \\ N, & N+1, & \ldots, & 2N-1 \end{array}$$

- Another canonical choice is to make $k = 0$ a center frequency

$$\begin{array}{rrrrrr} -N/2+1, & \ldots, & -1, & 0, & \ldots, & N/2 \\ N/2+1, & \ldots, & N-1, & N, & \ldots, & 3N/2 \end{array}$$

- With N even (as usual) we use $N/2$ positive frequencies and $N/2 - 1$ negative frequencies
- From one canonical set to the other $\Rightarrow$ Chop and shift

Theorem

Opposite frequencies k and $-k$ yield conjugate signals: $e_{-kN} = e_{kN}^*(n)$

Proof.

- Just use the definitions to write the chain of equalities

$$e_{-kN}(n) = \frac{e^{j2\pi(-k)n/N}}{\sqrt{N}} = \frac{e^{-j2\pi kn/N}}{\sqrt{N}} = \left[\frac{e^{j2\pi kn/N}}{\sqrt{N}}\right]^* = e_{kN}^*(n)$$

□

- Opposite frequencies $\Rightarrow$ Same real part. Opposite imaginary part

$\Rightarrow$ The cosine is the same, the sine changes sign

- Of N canonical frequencies, only $\dfrac{N}{2}+1$ are distinct. No more than $\dfrac{N}{2}$ oscillations in N samples

$$\begin{array}{rrrrr} 0, & 1, & \ldots, & N/2-1 & N/2 \\ & -1, & \ldots, & -N/2+1 & \\ & N-1, & \ldots, & N/2+1 & \end{array}$$

- The frequencies 0 and $N/2$ do not have a conjugate counterpart. All Others do

- The canonical set $-N/2+1, \ldots, -1, 0, 1, \ldots, N/2$ is easier to interpret

$\Rightarrow$ Positive frequencies ranging from 0 to $N/2 \leftrightarrow f_s/2$ have physical meaning

$\Rightarrow$ The negative frequencies are conjugates of the corresponding positive frequencies

Theorem

Complex exponentials with nonequivalent frequencies are orthogonal. I.e.

$$\langle e_{kN}, e_{lN} \rangle = 0$$

when $k - l < N$. *E.g., when* $k = 0, \ldots N-1$, *or* $k = -N/2+1, \ldots, N/2$.

- Signals of canonical sets are "unrelated." Different rates of change
- Also note that the energy is $\|e_{kN}\|^2 = \langle e_{kN}, e_{kN} \rangle = 1$
- Exponentials with frequencies $k = 0, 1, \ldots, N-1$ are orthonormal

$$\langle e_{kN}, e_{lN} \rangle = \delta(l-k)$$

- They are an orthonormal basis of signal space with N samples

Proof.

- Use definitions of inner product and discrete complex exponential to write

$$\langle e_{kN}, e_{lN} \rangle = \sum_{n=0}^{N-1} e_{kN}(n) e_{lN}^{*}(n) = \sum_{n=0}^{N-1} \frac{e^{j2\pi kn/N}}{\sqrt{N}} \frac{e^{-j2\pi ln/N}}{\sqrt{N}}$$

- Regroup terms to write as geometric series

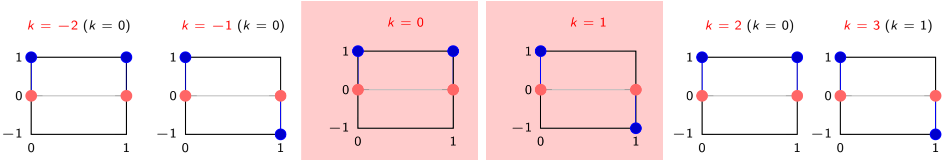
$$\langle e_{kN}, e_{lN} \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi (k-l)n/N} = \frac{1}{N} \sum_{n=0}^{N-1} \left[e^{j2\pi (k-l)/N} \right]^{n}$$

- Geometric series with basis a sums to $\sum_{n=0}^{N-1} a^n = (1-a^N)/(1-a)$. Thus,

$$\langle e_{kN}, e_{lN} \rangle = \frac{1}{N} \frac{1 - \left[e^{j2\pi (k-l)/N} \right]^{N}}{1 - e^{j2\pi (k-l)/N}} = \frac{1}{N} \frac{1-1}{1 - e^{j2\pi (k-l)/N}} = 0$$

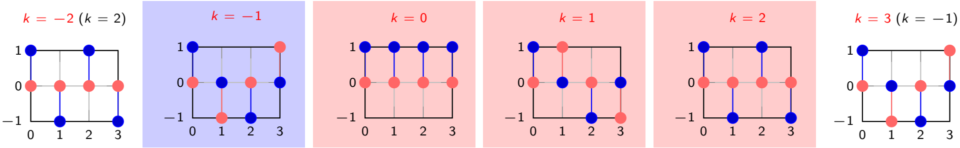
- Completed proof by noting $\left[e^{j2\pi (k-l)/N} \right]^{N} = e^{j2\pi (k-l)} = \left[e^{j2\pi} \right]^{(k-l)} = 1$ □

- When signal durations is $N = 2$ only frequencies $k = 0$ and $k = 1$ represent distinct signals



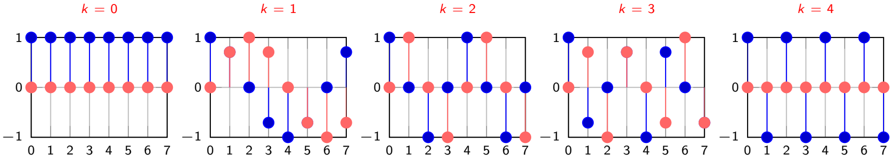
- The signals are real, they have no imaginary parts

- When $N = 4$, $k = 0, 1, 2$ are distinct. $k = -1$ is conjugate of $k = 1$

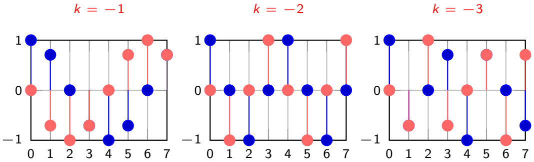


- Can also use $k = 3$ as canonical instead of $k = -1$ (conjugate of $k = 1$)

- Frequencies from $k = 1$ to $k = 4$ represent distinct signals



- Frequencies $k = -1$ to $k = -3$ are conjugate signals of $k = 1$ to $k = 3$



- All other frequencies represent one of the signals above

- There are 9 distinct frequencies and 7 conjugates (not shown). Some look like actual oscillations